ACKNOWLEDGMENT

J. A. Fergus Ross is grateful to Dr. F. Kschischang, University of Toronto, for suggesting [13], [14] and to the anonymous reviewer who suggested [5].

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Superimposed Codes for the Multiaccess Binary Adder Channel

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***Abstract*—Superimposed codes for multiple-access communication in a binary adder channel are analyzed. The superposition mechanism used in this correspondence is ordinary addition. Each user is assigned a codeword from a superimposed code. It is proved that every constant-weight code C of weight w and maximal correlation c corresponds to a subclass of a disjunctive code D of order $m < w/c$. Therefore, any m or less codewords in C which are used at the same time yield a uniquely decodable code combination at the output of the adder channel. In the noisy case, for each subset $A \subseteq C$ of size $|A| \leq m \ll T$ the receiver is able to determine the number of active users and to distinguish between the active users if the weight of the error pattern e satisfies $Wt(e) < \min\{w - c|A|, w/2\}$. Decoding algorithms for both the noiseless and the noisy cases are proposed.**

***Index Terms*— Superimposed codes, constant-weight codes, error-correcting codes, binary adder channel, multiaccess communication, information theory.**

I. INTRODUCTION

The idea of superimposed codes was introduced in 1964 by Kautz and Singleton [1]. The application they had in mind was information retrieval and the superposition mechanism assumed was a Boolean sum. The concept is, however, also useful in communications over the multiple-access OR channel. Many generalizations and results concerning the multiple-access OR channel have been obtained [2]–[4]. Chien and Frazer introduced the concept of superimposed codes by using modulo-2 addition as the superposition mechanism [5]; this was also recently reconsidered by Ericson and Levenshtein [6]. Ericson and Györfi studied the same problem in Euclidean n-space R^n in which the inputs and the output of the channel are all real-valued vectors [7]. In this correspondence we will further investigate superimposed codes. The superposition mechanism used here is ordinary addition. The application background of this superposition mechanism is also multiaccess communications, but the channel model used is a T-user binary adder channel (T-BAC) instead of the binary OR channel.

The multiaccess T-BAC, as shown in Fig. 1, is a multiple binary input single summed output channel [8]. By the summed output is meant the set of output codewords which results from the componentwise sum of codewords from a given code. The superimposed codes are binary codes and are characterized by three parameters: the block length n, the order m, and the size T. Identifying m out of T ($m \ll T$) users sharing a multiaccess adder channel can be achieved as follows: assign to each user one codeword from a superimposed code and the all-zero vector of length n. Those users who wish to identify themselves (active users) send their respective codewords from the superimposed code. All others send the zero vector. It is assumed that both block and bit synchronization are maintained. The code guarantees unique identification of all active users as long as the

Manuscript received August 16, 1993; revised December 9, 1994. The material in this correspondence was presented in part at the IEEE International Workshop on Information Theory, Moscow, Russia, July 3–8, 1994.

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IEEE Log Number 9411903.

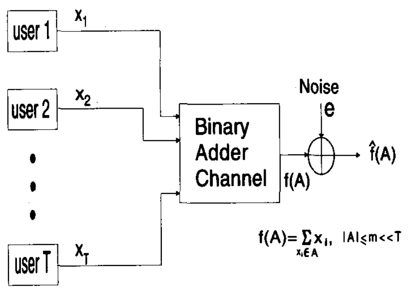

Fig. 1. Multiaccess binary adder channel.

number of active users does not exceed m. The decoder receives a sum vector, which is the superposition of the transmitted codewords, and it attempts to partition it into its component codewords.

In the following sections we will first present some basic concepts concerning the superposition mechanism and superimposed codes. Then the relationship between the constant-weight codes and disjunctive codes is analyzed and some important results concerning the decomposition of the disjunctive codes in the noiseless and noisy cases are derived. In order to demonstrate the operation of T-BAC system, a class of good constant-weight codes (KS codes) is discussed. Finally, several decoding algorithms for the noiseless and noisy channels are developed and the decoding complexity is briefly analyzed.

II. Basic Concepts

Before proceeding the following definitions are required:

Defintion 1: The correlation between two binary $\{0,1\}$ vectors x_l and x_k of length n is the number of positions in which both vectors are 1 (i.e., the number of overlaps between the two vectors)

$$c(x_l, x_k) = \sum_{j=0}^{n} x_{lj} \cdot x_{kj} \tag{1}$$

where the x_{lj}, x_{kj} are the jth binary symbols of x_l and x_k, respectively.

Given a binary code C, the maximum correlation c is defined as

$$c = \max_{x_l, x_k \in C,\ l \neq k} c(x_l, x_k). \tag{2}$$

Defintion 2: Consider a set $A = \{x_1, x_2, \cdots, x_m\}$ consisting of m binary vectors of length n. The superposition of these vectors is an m-ary vector $z = f(A) = (z_1, z_2, \cdots, z_n)$ of length n, where

$$z_j = \sum_{i=1}^{m} x_{ij} \quad j = 1, 2, \cdots, n. \tag{3}$$

This superposition concept corresponds to a binary adder channel that operates on a set A of binary input vectors and produces an output m-ary vector z equal to the ordinary sum of the input set.

Defintion 3: The weight of an m-ary vector $z = f(A) = (z_1, z_2, \cdots, z_n)$ is defined as

$$Wt(f(A)) = \sum_{j=1}^{n} |z_j|.$$

Let $x_i = \{x_{i1}, x_{i2}, \cdots, x_{in}\}$ be a binary vector. The weight of the difference vector $f(A) - x_i$ is defined as

$$Wt(f(A) - x_i) = \sum_{j=1}^{n} |z_j - x_{ij}| \tag{4}$$

where "$-$" denotes ordinary subtraction.

Definition 4: A binary vector $x_i = (x_{i1}, x_{i2}, \cdots, x_{in})$ is said to be included in an m-ary vector $z = (z_1, z_2, \cdots, z_n)$ if and only if $z_j - x_{ij} \geq 0, j = 1, 2, \cdots, n$.

Definition 5: The binary code C with codeword length n and size T is a disjunctive code of order m if each subset $A \subseteq C$ of size $|A| \leq m$ has the property that $z = f(A)$ includes only those codewords in C which are also in A. The set of all disjunctive codes with parameters n, m, and T is denoted by $D(n, m, T)$.

The class of disjunctive codes is a subset of the class of superimposed codes.

III. Constant-Weight Codes and Disjunctive Codes

Definition 6: A constant-weight (CW) code is a binary code in which all codewords have the same weight. For a CW code with weight w, the correlation is related to the Hamming distance d_H by

$$d_H(x_l, x_k) = 2w - 2c(x_l, x_k). \tag{5}$$

If we denote the minimum distance and maximum correlation by d_H and c, respectively, then

$$d_H = 2w - 2c. \tag{6}$$

The set of all CW codes with length n, weight w, maximum correlation c, and size T is denoted by $\mathrm{CW}(n, w, c, T)$.

Theorem 1: A binary CW code C with parameters (n, w, c, T) is also a disjunctive code $D(n, m, T)$, where m satisfying

$$m < \frac{w}{c}. \tag{7}$$

Proof: Let $A \subseteq C$, $A = \{x_1, x_2, \cdots, x_m\}$, and $x_c \in C$ be an arbitrary codeword not in A. Suppose $f(A)$ includes x_c, then $z_j - x_{cj} \geq 0, j = 1, 2, \cdots, n$. Because the code C has a maximum correlation c for all pairs of codewords, which means that each of the codewords $x_i \in A$ will overlap with x_c at most c times, then there are at most cm positions in x_c which will overlap with all $x_1, x_2, \cdots, x_m$. But from the conditions that the weight of x_c is a constant w and $cm < w$ (or $m < w/c$), it is obvious that there exist at least $w - cm$ positions that do not overlap with any of the codewords $x_i \in A$. In other words, there exist some positions j such that $z_j - x_{cj} < 0$, which implies x_c cannot be included in $z = f(A)$ if $m < w/c$. From definition (5) we can thus conclude that a binary CW code C with parameters (n, w, c, T) is also a disjunctive code $D(n, m, T)$.

In practice, the order m of a disjunctive code $D(n, m, T)$ should be an integer. In order to use Theorem 1, we can set

$$m = \left\lceil \frac{w}{c} \right\rceil - 1 < \frac{w}{c} \tag{8}$$

where $\lceil x \rceil$ denotes the lowest integer greater than or equal to x. This relation is useful because it transforms the problem of designing disjunctive codes into the problem of designing constant-weight codes which have been extensively studied in the past [9], [10]. It should be pointed out that the constant-weight codes correspond to only a subset of disjunctive codes. There may be good disjunctive codes that are not constant weight codes. ■

Theorem 2: If the binary code C is a disjunctive code $D(n, m, T)$ constructed from $\mathrm{CW}(n, w, c, T)$, then for each subset $A \subseteq C$ of size $|A| \leq m$, the equation $Wt(f(A) - x_i) = (|A| - 1)w$ holds when $x_i \in A$. But for all other codewords $x_c \in C \setminus A$, we have $\mathrm{Wt}(z - x_c) > (|A| - 1)w$.

Proof: If the binary code C is a constant weight code $\mathrm{CW}(n, w, c, T)$ then for $A \subseteq C$, it is simple to show that the weight of $z = f(A)$ is equal to

$$\mathrm{Wt}(z) = \sum_{\{i|x_i \in A\}} \sum_{j=1}^{n} |x_{ij}| = |A|w. \tag{9}$$

Because $\mathrm{Wt}(x_i) = w$, if $x_i \in A$ ($f(A)$ includes x_i) then

$$\mathrm{Wt}(f(A) - x_i) = \sum_{j=1}^{n} |z_j - x_{ij}| = |A|w - w = (|A| - 1)w \tag{10}$$

which proves the first part of the theorem.

If $x_c \in C \setminus A$ then $f(A)$ does not include x_c. It has been shown in the proof of Theorem 1 that there exist at least $w - cm$ positions that do not overlap with any of the codewords $x_i \in A$, i.e., $z_j - x_{cj} < 0$ for some j. Let S denote the set of positions of nonzero elements in x_c which overlap with other $x_i \in A$ and $\bar{S}$ denote the set of positions of nonzero elements x_{cj} which do not overlap with any of $x_i \in A$. Because $(w - cm) > 0$ and $|A| \leq m$, we have

$$\begin{aligned}\mathrm{Wt}(f(A) - x_c) &= \sum_{j \in S} |z_j - x_{cj}| + \sum_{j \in \bar{S}} |z_j - x_{cj}| \\ &\geq (|A|w - c|A|) + (w - c|A|) \\ &= (|A| - 1)w + 2(w - c|A|) \\ &> (|A| - 1)w \end{aligned} \tag{11}$$

which concludes the proof of the theorem. ■

Theorem 3: Let C be a binary disjunctive code $D(n, m, T)$ constructed from $\mathrm{CW}(n, w, c, T)$, then for each subset $A \subseteq C$ of size $|A| \leq m$ the receiver is able to distinguish how many codewords have been transmitted if the weight of the error pattern $e = (e_1, e_2, \cdots, e_n)$ satisfies

$$\mathrm{Wt}(e) = \sum_{j=1}^{n} |e_j| < w/2.$$

Proof: From Theorem 1, if $|A| = m' \leq m$, $\mathrm{Wt}(f(A)) = m'w$; if $|A| = m' + 1 \leq m$, then $\mathrm{Wt}(f(A)) = (m' + 1)w$. Let

$$S = \mathrm{Wt}(\hat{f}(A)) = \mathrm{Wt}(f(A) + e)$$

then

$$\mathrm{Wt}(f(A)) - \mathrm{Wt}(e) \leq S \leq \mathrm{Wt}(f(A)) + \mathrm{Wt}(e) \tag{12}$$

or

$$|A|w - \mathrm{Wt}(e) \leq S \leq |A|w + \mathrm{Wt}(e). \tag{13}$$

Now if $\mathrm{Wt}(e) < w/2$, the number of codewords transmitted can be obtained by

$$|A| = \begin{cases} \lfloor \frac{S}{w} \rfloor & (S - \lfloor \frac{S}{w} \rfloor w) < \frac{w}{2} \\ \lfloor \frac{S}{w} \rfloor + 1 & (S - \lfloor \frac{S}{w} \rfloor w) \geq \frac{w}{2} \end{cases} \tag{14}$$

where $\lfloor x \rfloor$ denotes the highest integer less than or equal to x. ■

Theorem 4: Let C be a binary disjunctive code $D(n, m, T)$ constructed from $\mathrm{CW}(n, w, c, T)$, then for each subset $A \subseteq C$ of size $|A| \leq m$ the receiver is able to correct any error pattern $e = (e_1, e_2, \cdots, e_n)$ whose weight satisfies

$$\mathrm{Wt}(e) = \sum_{j=1}^{n} |e_j| < w - c|A|.$$

Proof: Suppose the received vector is $\hat{f}(A) = f(A) + e$. For each subset $A \subseteq C$ of size $|A| \leq m$, if the channel is noiseless, then

$$\mathrm{Wt}(\hat{f}(A) - x_i) = (|A| - 1)w, \quad x_i \in A.$$

If the channel is noisy and the weight of the error pattern satisfies $\mathrm{Wt}(e) < w - c|A|$, let $x_i \in A$, then

$$\begin{aligned}\mathrm{Wt}(\hat{f}(A) - x_i) &= \mathrm{Wt}(f(A) + e - x_i) \\ &= \sum_{j \in S} |z_j + e_j - x_{ij}| \\ &\quad + \sum_{j \in \bar{S}} |z_j + e_j - x_{ij}| \\ &\leq (|A|w - w) + \mathrm{Wt}(e) \\ &< (|A| - 1)w + (w - c|A|). \end{aligned} \tag{15}$$

But for all other codewords $x_c \in C \setminus A$ and $\mathrm{Wt}(e) < w - c|A|$, we have

$$\begin{aligned}\mathrm{Wt}(\hat{f}(A) - x_c) &= \mathrm{Wt}(f(A) + e - x_c) \\ &= \sum_{j \in S} |z_j + e_j - x_{cj}| \\ &\quad + \sum_{j \in \bar{S}} |z_j + e_j - x_{cj}| \\ &\geq (|A|w - c|A|) + (w - c|A|) - \mathrm{Wt}(e) \\ &= (|A| - 1)w + 2(w - c|A|) - \mathrm{Wt}(e) \\ &> (|A| - 1)w + (w - c|A|). \end{aligned} \tag{16}$$

Therefore, the correct codewords $x_i \in C$ can be distinguished from the codewords $x_c \in C \setminus A$.

It should be noted that the error-correcting ability of the disjunctive code constructed from the constant-weight code is not constant, but is a function of size $|A|$. It can be seen from the condition, $\mathrm{Wt}(e) < w - c|A|$, that the maximum and minimum error weights that can be corrected are, respectively, $\mathrm{Wt}_{\max}(e) = w - c = d_H/2$ (when $|A| = 1$) and $\mathrm{Wt}_{\min}(e) = w - c|m|$(when $|A| = m$), where $d_H = 2w - 2c$ is the minimum Hamming distance of the constant-weight code. It is clear that in order to use Theorem 4, the size of A should be known in advance. Although the $|A|$ can be obtained from (14), Theorem 3 can only guarantee the correctness of the $|A|$ computed by (14) if

$$Wt(e) = \sum_{j=1}^{n} |e_j| < w/2.$$

This analysis directly leads to Theorem 5 below: ■

Theorem 5: Let C be a binary disjunctive code $D(n, m, T)$ constructed from $\mathrm{CW}(n, w, c, T)$, then for each subset $A \subseteq C$ of size $|A| \leq m$ the receiver is able to distinguish how many codewords have been transmitted and, at the same time, recover every codeword if the weight of the error pattern e satisfies

$$\mathrm{Wt}(e) = \sum_{j=1}^{n} |e_j| < \min\left\{w - c|A|, \frac{w}{2}\right\}. \tag{17}$$

IV. Concatenated KS Code

An effective method for constructing good constant-weight codes (and thereby disjunctive codes) is to use a concatenated code in which the inner code is a constant-weight code. The KS code, which was first found by Kautz and Singleton [1], is based on a maximum-distance-separable (MDS) outer code (e.g., Reed–Solomon code) and an orthogonal weight-one inner code $\mathrm{CW}(q, 1, 0, q)$. Let C be an RS code over $\mathrm{GF}(q)$ with length w and minimum distance d. The dimension will be $k = w - d + 1$. We now produce the inner

$\mathrm{CW}(q, 1, 0, q)$ code by mapping each symbol in $\mathrm{GF}(q)$ to a binary vector of length q and weight one

$$\begin{aligned} 0 &\longrightarrow 1000\cdots0 \\ 1 &\longrightarrow 0100\cdots0 \\ &\cdots \\ (q-1) &\longrightarrow 0000\cdots1. \end{aligned}$$

We, therefore, obtain a constant-weight code $\mathrm{CW}(qw, w, c, q^{c+1})$ which corresponds to a disjunctive code $D(wq, \lceil w/c \rceil - 1, q^{c+1})$.

Example: Let C be an $\mathrm{RS}(n, k, d) = \mathrm{RS}(6, 2, 5)$ code with generator matrix

$$G = \begin{bmatrix} 1 & 0 & 6 & 5 & 4 & 3 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}. \tag{18}$$

With this RS code, a disjunctive code with parameters

$$D\Big(6 \times 7, \left\lceil \frac{6}{6-5} \right\rceil - 1, 7^{6-5+1}\Big) = D(42, 5, 49) \tag{19}$$

and Hamming distance $d_H = 10$ can be constructed.

Any RS codeword $x_i = (x_{i1}, x_{i2}, \cdots, x_{i6})$, $x_{ij} \in \mathrm{GF}(7)$ can be calculated by multiplying the generator matrix by an information vector $m = (m_1, m_2)$, $m_j \in \mathrm{GF}(7)$. For example, if $m \in \{(1,2), (2,2), (3,2), (6,6)\}$, then the codeword set A, $|A| \le 5$, and the superposition of A will be as follows:

$$\begin{aligned} A &= \{mG \mid m \in \{(1,2),(2,2),(3,2),(6,6)\}\} \\ &= \{(123456), (222222), (321065), (666666)\} \\ &= \left\{ \begin{pmatrix} 000000 \\ 100000 \\ 010000 \\ 001000 \\ 000100 \\ 000010 \\ 000001 \end{pmatrix}, \begin{pmatrix} 000000 \\ 000000 \\ 111111 \\ 000000 \\ 000000 \\ 000000 \\ 000000 \end{pmatrix}, \begin{pmatrix} 000100 \\ 001000 \\ 010000 \\ 100000 \\ 000000 \\ 000001 \\ 000010 \end{pmatrix}, \begin{pmatrix} 000000 \\ 000000 \\ 000000 \\ 000000 \\ 000000 \\ 000000 \\ 111111 \end{pmatrix} \right\} \end{aligned} \tag{20}$$

$$z = f(A) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 \end{pmatrix}. \tag{21}$$

V. Decoding Algorithms for the Noiseless T-BAC

The objective of the decoder for a superimposed code is to reproduce the transmitted codewords of the active users using the received vector. For the noiseless T-BAC, the task is just to map from the received superimposed m-ary vector $f(A)$ formed by the channel into a set of codewords $\hat{A}$ from the given superimposed code C. That is

$$f(A) \longrightarrow \hat{A}. \tag{22}$$

Decoding Algorithm 1: According to Theorem 2, if the binary code C is a binary disjunctive code $D(n, m, T)$ constructed from $\mathrm{CW}(n, w, c, T)$, then for each subset $A \subseteq C$ of size $|A| \le m$, $\mathrm{Wt}(f(A) - x_i) = (|A| - 1)w$ if and only if $x_i \in A$. Therefore, an obvious way of decoding is an exhaustive search, i.e., for all codewords in $x_i \in C$, compute the $\mathrm{Wt}(f(A) - x_i)$, and then output all the codewords which satisfy $\mathrm{Wt}(f(A) - x_i) = (|A|-1)w$. That is

$$\hat{A} = \{x_i \in C \mid \mathrm{Wt}(f(A) - x_i) = (|A| - 1)w\} \tag{23}$$

where $|A|$ is obtained by (14).

Decoding Algorithm 2: The exhaustive search decoder has a decoding complexity independent of the transmitted set of codewords and equal to T. If we make use of the structure of the specific disjunctive code, the number of codewords x_i used in the test $\mathrm{Wt}(f(A) - x_i) = (|A|-1)w$ can be reduced greatly and the decoding complexity thus reduced. In the case of a KS code, it is simple to find the transmitted elements (symbols) from $\mathrm{GF}(q)$ in the received vector. In the above example, from the first and second columns of the received superposition vector $z = f(A)$, it is evident that the first and second positions of the transmitted RS codewords must be in the set $\{1, 2, 3, 6\}$ and $\{2, 6\}$, respectively. This means that the possible information vectors $m = (m_1, m_2)$ of the corresponding codewords are

$$m \in \{(1,2), (1,6), (2,2), (2,6), (3,2), (3,6), (6,2), (6,6)\}. \tag{24}$$

Obviously, if we use the corresponding codeword set $\dot{C}$ as a candidate set in the decoding process, i.e.

$$\hat{A} = \Big\{x_i \in \dot{C} \mid \mathrm{Wt}(f(A) - x_i) = (|A| - 1)w\Big\} \tag{25}$$

the results will be the same as with Algorithm 1. However, because $|\dot{C}| \ll |C|$, the decoding complexity has been reduced greatly. For the example given above, $|C| = 49$, but $|\dot{C}| = 8$.

Based on the fact that the size of $|A|$ is constant for a given received vector $f(A)$ and can be calculated beforehand, the decoding complexity can be further reduced by counting the number of decoded codewords. If the number of decoded codewords equals $|A|$, then there is no need to try the rest of the candidate codewords.

Therefore, the simplified algorithm can be summarized as follows:

1) Compute the size of $|A|$ using (14).
2) Generate a relatively small candidate codeword set $\dot{C}$ by making use of the specific code structure.
3) Select a candidate codeword $x_i \in \dot{C}$ and test whether it satisfies the following condition:
$$\mathrm{Wt}(f(A) - x_i) = (|A| - 1)w. \tag{26}$$
If it does, increase the decoded codeword counter by one.
4) If the counter value is equal to $|A|$, then exit; otherwise go to Step 3.

VI. Decoding Algorithms for the Noisy Case

As is shown in Fig. 1, the received vector in the noisy case is given by $\hat{f}(A) = f(A) + e$. Based on Theorem 5, if C is a binary disjunctive code $D(n, m, T)$ constructed from $\mathrm{CW}(n, w, c, T)$ and the weight of the error pattern e satisfies

$$\mathrm{Wt}(e) = \sum_{j=1}^{n} |e_j| < \min\{w - c|A| \frac{w}{2}\}, \quad |A| \le m$$

then the codewords transmitted can be correctly recovered. Therefore, we have the following decoding algorithm:

Decoding Algorithm 3:

$$\hat{A} = \Big\{x_i \in C \mid \mathrm{Wt}(\hat{f}(A) - x_i) < (|A| - 1)w + \min\Big\{w - c|A|, \frac{w}{2}\Big\}\Big\}. \tag{27}$$

In order to reduce the decoding complexity, the same idea can be employed as in Algorithm 2. That is, to produce a relatively small set of candidate codewords $\ddot{C}$ by making use of the structure of the specific disjunctive code and the known information $|A|$. For the KS code, however, the size of the candidate set $\ddot{C}$ and the process of producing candidate codewords will be slightly different; this is not addressed here, for simplicity. After generating the candidate set $\ddot{C}$, the decoding process is similar, i.e.:

Decoding Algorithm 4:

$$\hat{A} = \Big\{x_i \in \ddot{C} \mid \mathrm{Wt}(\hat{f}(A) - x_i) < (|A| - 1)w + \min\Big\{w - c|A|, \frac{w}{2}\Big\}\Big\} \quad (28)$$

where the decoding process should be stopped if the decoded codeword counter is equal to $|A|$.

Finally, it should be stressed that the error pattern e is not necessarily an integer vector; it can also be any real value vector if

$$\mathrm{Wt}(e) < \min\Big\{w - c|A|, \frac{w}{2}\Big\}.$$

In other words, the proposed Algorithms 3 and 4 are also soft-decision decoding algorithms which will give better error performance than with hard-decision decoding.

VII. Summary

In this paper, we have investigated superimposed codes for the T-BAC. The superposition mechanism used here is ordinary addition. The T-BAC system consists of a set of T users sharing a multiaccess binary adder channel. It has been proved that if the number $|A|$ of active users satisfies the condition $|A| \leq m \ll T$, we can decompose the received word into its component codewords over a noiseless T-BAC. In the noisy case, the number of active users and the codewords can also be correctly recovered provided that the weight of the error pattern satisfies

$$\mathrm{Wt}(e) = \sum_{j=1}^{n} |e_j| < \min\Big\{w - c|A|, \frac{w}{2}\Big\}.$$

In this paper, each user is given only one codeword, which can only be used to identify the active users. However, if each user is distributed a set of codewords, information can be carried and transmitted.

Acknowledgment

The authors wish to thank an anonymous referee for his helpful suggestions.

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New Optimal Ternary Linear Codes

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***Abstract*—The class of quasi-twisted (QT) codes is a generalization of the class of quasi-cyclic codes, similar to the way constacyclic codes are a generalization of cyclic codes. In this paper, rate $1/p$ QT codes over GF(3) are presented which have been constructed using integer linear programming and heuristic combinatorial optimization. Many of these attain the maximum possible minimum distance for any linear code with the given parameters, and several improve the maximum known minimum distances. Two of these new codes, namely $(90, 6, 57)$ and $(120, 6, 78)$, are optimal and so prove that $d_3(90, 6) = 57$ and $d_3(120, 6) = 78$.**

***Index Terms*—Quasi-twisted codes, heuristic search, optimal codes over GF(3).**

I. Introduction

One of the most fundamental and challenging problems in coding theory is to construct a linear (n, k) code over GF(q) achieving the maximum possible (or known) minimum Hamming distance, d. This value is denoted as $d_q(n, k)$, and linear codes which achieve it are *optimal.* The Gilbert–Varshamov bound [1] gives a lower bound on $d_q(n, k)$, but few classes of codes are known which attain this bound. One exception is the class of rate $1/p$ quasi-twisted (QT) codes, which has been shown to meet this bound [2]. Therefore, it is not surprising that good QT codes exist for many values of $d_q(n, k)$.

QT codes were first characterized by Hill and Greenough [3]. They are a generalization of the class of quasi-cyclic (QC) codes in the same way that constacyclic codes are a generalization of cyclic codes [4], [5]. In this correspondence, only the subclass of rate $1/p$ QT codes constructed from $m \times m$ twistulant matrices is considered.

A *best* QT code is defined as one which achieves the maximum possible minimum distance for a QT code. A *good* code is defined as one which has the maximum known minimum distance, i.e., it attains (or improves) the known lower bound on the minimum distance. An exhaustive search for a best code (using, say, integer linear programming) is intractable for all but the smallest code dimensions. Heuristic techniques provide a means of constructing good codes with a reasonable amount of computational effort. The quality of a code constructed in this manner can be determined by comparing it with a known bound. Lower bounds for linear codes over GF(3) for $k \leq n \leq 50$, have been tabulated by Kschischang and Pasupathy [5], and an improved table of both lower and upper bounds for these dimensions has recently been constructed by Daskalov, Hill, and

Manuscript received July 26, 1994; revised December 12, 1994. This work was supported in part by the Natural Sciences and Engineering Research Council of Canada.

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IEEE Log Number 9412248.