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Construction of a superimposed code using partitions

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Abstract

This paper will focus on the construction of superimposed codes using incidence matrices. Such constructions require a set of elements and a partial order defined on the set. We will define a partial order on partitions. The construction will be made using elements from the partially ordered set of partitions of n elements.

Keywords: Coding theory; Superimposed code; Partitions

1. Definitions

Definition 1. Consider the following matrix:

$$X = \begin{bmatrix} x_{11} & x_{12} & . & . & . & x_{1t} \\ x_{21} & & & & & \\ . & & & x_{ij} & & \\ . & & & & & \\ . & & & & & \\ x_{N1} & x_{N2} & . & . & . & x_{Nt} \end{bmatrix}, \quad x_{ij} \in \{0, 1\}.$$

The above $N \times t$ matrix X will be referred to as a *code*. The columns of X are the *codewords*. Let $\mathbf{x}_j$ denote the jth codeword. Then we have a code of size t and length N. Notice that code X is a collection of codewords that are represented by binary vectors; hence we can use the concept of a *Boolean sum* and *intersection* of two binary vectors.

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Definition 2. The Boolean sum of two binary codewords is denoted by $\mathbf{x}_i \vee \mathbf{x}_j = (x_{1i} \vee x_{1j}, \ldots, x_{Ni} \vee x_{Nj})$, where $\vee$ is the OR operation.

Definition 3. The intersection vector of two binary codewords is denoted by $\mathbf{x}_i \wedge \mathbf{x}_j = (x_{1i} \wedge x_{1j}, \ldots, x_{Ni} \wedge x_{Nj})$, where $\wedge$ is the AND operation.

Definition 4. We say that $\mathbf{x}_i$ *covers* $\mathbf{x}_j$ if $\mathbf{x}_i \vee \mathbf{x}_j = \mathbf{x}_i$.

Definition 5. A code X has strength s if the Boolean sum of any s codewords does not cover any other codeword in X. A code that has a strength s is an (N, s, t) *superimposed* code.

Definition 6. The *weight* $w(\mathbf{x}_i)$ of binary codeword $\mathbf{x}_i$ is the number of nonzero elements in the codeword. Let $w = \min_{1 \leqslant i \leqslant t} w(\mathbf{x}_i)$.

Definition 7. The *intersection* $\lambda(\mathbf{x}_i, \mathbf{x}_j)$ between two codewords—$\mathbf{x}_i$, $\mathbf{x}_j$—is the number of places in which both $\mathbf{x}_i$ and $\mathbf{x}_j$ have nonzero elements, i.e. $\lambda(\mathbf{x}_i, \mathbf{x}_j) = w(\mathbf{x}_i \wedge \mathbf{x}_j)$. Let

$$\lambda = \max_{\substack{1 \leqslant i,j \leqslant t \\ i \neq j}} \lambda(\mathbf{x}_i, \mathbf{x}_j).$$

Theorem (*Kautz–Singleton*). *A lower bound for the value of the parameter s is* [3]: $s \geqslant \lfloor (w-1)/\lambda \rfloor$.

Remark. Obviously if we have that any s codewords do not cover any $\mathbf{x}_j$, then we also have any number of codewords less than s will also not cover $\mathbf{x}_j$.

Let $n, q_1, q_2 \in \mathbb{N}$, where $2 \leqslant q_1 \leqslant q_2 \leqslant n$. $A_{q_i} \equiv \{0, 1, \ldots, q_i - 1\}$ is the standard q_i-ary alphabet, $i \in \{1, 2\}$, and $[n] = \{1, 2, \ldots, n\}$ is an ordered set of n elements. $M_{q_i} \equiv \{\mu_1, \mu_2, \ldots, \mu_{q_i!}\}$ is the set of all $q_i!$ permutations of q_i symbols. Let $\mathbf{y} \equiv (y_1, y_2, \ldots, y_n)$, $y_i \in A_{q_i}$ denote an arbitrary q_i-ary n-sequence that identifies an unordered q_i*-partition* $\{E_0; E_1; \ldots; E_{q_i}\}$ of $[n]$ where $E_m = \{i : y_i = m\}$, $m \in A_{q_i}$ e.g. if $[n] = \{1, 2, 3, 4, 5\}$ and $q_i = 3$; then $\mathbf{y} = (1, 1, 1, 2, 0)$ identifies the partition $\{(5); (1, 2, 3); (4)\}$ where $E_0 = \{5\}$, $E_1 = \{1, 2, 3\}$ and $E_2 = \{4\}$.

Remark. Any q_i-partition contains q_i', $1 \leqslant q_i' \leqslant q_i$, nonempty *clusters*.

For any $\mu_i \in M_{q_i}$ we can identify a q_i-ary n-sequence: $\mathbf{y}^{\mu_i} = (\mu_i(y_1), \mu_i(y_2), \ldots, \mu_i(y_n))$ called a μ-complement of $\mathbf{y}$. Notice that all μ-complements of any $\mathbf{y}$ identify the same unordered q_i-partition. In our construction we want to isolate all partitions of a set that have q_1 and q_2 nonempty clusters. Then we will define a partial order relation between them. Let us denote a partition with q_i nonempty clusters by $\widetilde{\mathbf{y}}_{q_i}$. The set of all partitions of $[n]$ that contain q_i nonempty clusters will be denoted by $S_{q_i}(n)$. In addition to μ-complement where a bijection acts on a vector, we would also like to introduce this operation but with a surjection from A_{q_2} onto A_{q_1}. We will say that vector $\mathbf{y}$ is mapped by ϕ to $\mathbf{y}^{\phi} = (\phi(y_1), \phi(y_2), \ldots, \phi(y_n))$. Let us introduce the following relation $\succ$. We will say that

$\widetilde{\mathbf{y}}_{q_1} \succ \widetilde{\mathbf{z}}_{q_2}$ iff $\exists\phi : A_{q_2} \to A_{q_1}$, a surjection, such that $\mathbf{z}_{q_2}^{\phi} = \mathbf{y}_{q_1}$ for some vector $\mathbf{z}_{q_2}$ corresponding to $\widetilde{\mathbf{z}}_{q_2}$, and $\mathbf{y}_{q_1}$ corresponding to $\widetilde{\mathbf{y}}_{q_1}$. $\widetilde{\mathbf{z}}_{q_2}$ is then called a *sub-partition* of $\widetilde{\mathbf{y}}_{q_1}$. Notice that if $q_1 < q_2$, then we can never have $\widetilde{\mathbf{z}}_{q_2} \succ \widetilde{\mathbf{y}}_{q_1}$ since we cannot define a surjection from one set to a larger set.

We would now like to find the number of partitions that have q_1 clusters (nonempty parts) and also the number of partitions that have q_2 clusters. This is a problem solved by using Stirling set numbers of the second kind [4]. We have that the number of partitions of $[n]$ having q_i clusters is

$$|S_{q_i}(n)| = \frac{1}{q_i!}\sum_{j=0}^{q_i-1}(-1)^j\binom{q_i}{j}(q_i - j)^n.$$

2. Construction

The construction of this superimposed code is similar to the construction used in [1] with the exception that we use the Kautz–Singleton bound for the strength s. Instead of using the set–subset relation, we shall use the partition–subpartition relation as defined above. Let us denote elements from $S_{q_2}(n)$ by P_i, $1 \leqslant i \leqslant |S_{q_2}(n)|$ and elements from $S_{q_1}(n)$ by R_j, $1 \leqslant j \leqslant |S_{q_1}(n)|$. Consider the following matrix:

$$\begin{matrix} & [P_1 & & \dots & P_i & \dots & P_t] \\ \begin{bmatrix} R_1 \\ \cdot \\ \cdot \\ R_j \\ \cdot \\ R_N \end{bmatrix} & \begin{bmatrix} x_{11} & x_{12} & \cdot & \cdot & \cdot & x_{1t} \\ x_{21} & & & & & \\ \cdot & & & & & \\ \cdot & & & x_{ji} & & \\ \cdot & & & & & \\ x_{N1} & x_{N2} & \cdot & \cdot & \cdot & x_{Nt} \end{bmatrix} \end{matrix},$$

where $x_{ji} = 1$ iff $R_j \succ P_i$ else $x_{ji} = 0$. Notice that for this code we have

$$t = |S_{q_2}(n)|,$$
$$N = |S_{q_1}(n)|.$$

We can also find the weight and maximum intersection and hence a value for s—the strength of the code. First of all notice that to find the weight of any column i we need to find the number of surjections from P_i to partitions containing q_1 clusters. This is translated to being the number of surjections from A_{q_2} to A_{q_1}. This value depends only on the values q_1 and q_2, which means that as we increase n and hence increase the number of codewords t and their length N, we will always have a constant weight (assuming we keep q_1 and q_2 constant). The number of surjections from a set of size q_2 to a set of size q_1 is a well-known formula:

$$\sum_{i=0}^{q_1-1}(-1)^i\binom{q_1}{i}(q_1 - i)^{q_2}.$$

Now we have to notice that for any surjection, a permutation composed with the surjection is also a surjection (a different one) but the partition to which it is mapping its subject to is the same because the vector is simply a μ-complement. More formally,

$$\mathbf{y}_{q_2}^{\phi} \approx \widetilde{\mathbf{z}}_{q_1},$$

$$(\mathbf{y}_{q_2}^{\phi})^{\mu_i} \approx \widetilde{\mathbf{z}}_{q_1}, \quad \forall \mu_i \in M_{q_1}.$$

Also $\forall \mu_i \in M_{q_1}$, $\mu_i \neq (1)$, $\mu_i \circ \phi = \varphi \neq \phi$, and since the composition of an onto mapping with an onto mapping is again onto, we have that φ is onto. The weight is going to be the number of surjections from a given partition P_i to *distinct* partitions. Since the vectors representing partitions of the form R_j contain all q_1 symbols, we must divide the sum of surjections by $q_1!$ to obtain w:

$$w = \frac{1}{q_{1!}} \sum_{i=0}^{q_1-1} (-1)^i \binom{q_1}{i} (q_1 - i)^{q_2} = |S_{q_1}(q_2)|.$$

Note that the value for w is a Stirling set number, i.e. it is the number of ways of partitioning a set of q_2 elements into q_1 clusters, which is exactly what we are doing in the partial order relation.

To find λ, the maximum intersection, we must first specify a *partition distance* as used in [2]:

Definition 8. The partition distance between two partitions P_i, P_j is $d_P(P_i, P_j) = \min_{\mu \in M_{q_2}} d_H(\mathbf{y}_{q_2}, \mathbf{z}_{q_2}^{\mu})$, where d_H is the Hamming metric and $\mathbf{y}_{q_2} \approx P_i$, $\mathbf{z}_{q_2} \approx P_j$.

Lemma. *If $d_P(P_i, P_j) = 1$, then $\lambda(\mathbf{x}_i, \mathbf{x}_j) = \lambda$.*

Proof. Let P_i, P_j be partitions, containing q_2 nonempty clusters, such that $d_P(P_i, P_j) = 1$. By definition, there exist vectors $\mathbf{y} \approx P_i$, $\mathbf{z} \approx P_j$ such that $d_H(\mathbf{y}, \mathbf{z}) = 1$. This means that $\exists i$, $1 \leqslant i \leqslant t$, $y_i = a \neq z_i = b$. To obtain the number of partitions from $S_{q_1}(n)$ that have P_i and P_j as sub-partitions, we group clusters a and b into one cluster to obtain a partition containing $q_2 - 1$ nonempty clusters. We then find that the number of surjections from P_i *and* P_j into $S_{q_1}(n)$ is $|S_{q_1}(q_2 - 1)|$. Hence $\lambda(\mathbf{x}_i, \mathbf{x}_j) = |S_{q_1}(q_2 - 1)|$. If on the other hand, we have that $d_P(P_i, P_j) > 1$, then we will also have $d_H(\mathbf{y}, \mathbf{z}) > 1$, meaning that there will be more positions in the vectors that will have differing elements. In the same way, to find $\lambda(\mathbf{x}_i, \mathbf{x}_j)$, we would have to group clusters together. This could produce partitions that have $q_2 - k$, $k = \overrightarrow{2, q_2 - 1}$ nonempty clusters, producing an intersection:$\lambda(\mathbf{x}_i, \mathbf{x}_j) = |S_{q_1}(q_2 - k)|$. Stirling set numbers of the second kind decrease as the size of the sequence being partitioned decreases and the number of clusters the sequence is partitioned into remains the same, i.e. $|S_{q_1}(q_2 - k)| < |S_{q_1}(q_2 - 1)|$, $k = \overrightarrow{2, q_2 - 1}$. This means that a case in which the maximum intersection λ occurs is between partitions that are at minimal distance from each other and it is equal to $|S_{q_1}(q_2 - 1)|$. $\square$

Table 1

n	t	N	q_2	q_1	s
10	34 105	9330	4	3	5
10	42 525	511	5	2	2
10	42 525	9330	5	3	4
10	42 525	34 105	5	4	9
10	22 827	34 105	6	4	6
10	22 827	42 525	6	5	14
20	45 232 115 901	524 287	4	2	2
20	45 232 115 901	580 606 446	4	3	5
20	749 206 090 500	524 287	5	2	2
20	749 206 090 500	580 606 446	5	3	4
20	749 206 090 500	45 232 115 901	5	4	9

Now we can calculate s using the Kautz–Singleton bound:

$$s = \left\lfloor \frac{w-1}{\lambda} \right\rfloor = \left\lfloor \frac{|S_{q_1}(q_2)|-1}{|S_{q_1}(q_2-1)|} \right\rfloor.$$

Below is a summary of the parameters for the code that we constructed.

$$t = |S_{q_2}(n)|, \quad w = |S_{q_1}(q_2)|,$$

$$N = |S_{q_1}(n)|, \quad \lambda = |S_{q_1}(q_2-1)|.$$

Table 1 gives a list of some parameters that were found using specific values for n, q_1, q_2 and using the Kautz–Singleton bound.

References

[1] A. D'yachkov, A. Macula, D. Torney, P. Vilenkin, S. Yekhanin, New results in the theory of superimposed codes, (in two parts), in: Proceedings of ACCT-7, Bansko, Bulgaria, 2000, pp. 126–136.

[2] A. D'yachkov, V. Rykov, D. Torney, S. Yekhanin, Partition codes, in: Proceedings of IEEE International Symposium on Information Theory, Chicago, USA, 2004, pp. 256.

[3] W.H. Kautz, R.C. Singleton, Nonrandom binary superimposed codes, IEEE Trans. Inform. Theory 4 (1964) 363–377.

[4] J.H. van Lint, R.M. Wilson. A Course in Combinatorics, Cambridge, 2001.