Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Discrete Applied Mathematics 128 (2003) 85–101
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

Some constructions of superimposed codes in Euclidean spaces

Danyo Danev

Department of Electrical Engineering, Linköping University, 581 83 Linköping, Sweden

Received 8 March 2001; received in revised form 20 August 2001; accepted 8 April 2002

Abstract

We describe three new methods for obtaining superimposed codes in Euclidean spaces. With help of them we construct codes with parameters improving upon known constructions. We also prove that the spherical simplex code is not optimal as superimposed code at least for dimensions greater than 9.

Keywords: Superimposed codes; Spherical codes; Euclidean channel; Multiple-access communications

1. Spherical superimposed codes

In [7,8], a class of superimposed codes for the Euclidean channel was introduced. In this paper, we consider in detail this type of codes. As usual we shall denote the n-dimensional Euclidean space by $\mathbb{R}^n$ and its elements will be called *vectors* or *points*. The standard inner product (sometimes called *correlation*) between two vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle \triangleq \sum_{i=1}^{n} x_i y_i.$$

The *Euclidean norm* of the vector $\mathbf{x}$ is $\|\mathbf{x}\|_E \triangleq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}$. The distance between $\mathbf{x}$ and $\mathbf{y}$ is $d_E(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_E$. The induced metric is called *Euclidean metric*.

E-mail address: danyo@isy.liu.se (D. Danev).

0166-218X/03/$ - see front matter
PII: S0166-218X(02)00438-9

For an arbitrary finite subset $A = \{\mathbf{a^{(1)}}, \mathbf{a^{(2)}}, \dots, \mathbf{a^{(T)}}\}$ of $\mathbb{R}^n$ the minimum distance $d_{\mathrm{E}}(A)$ of A is

$$d_{\mathrm{E}}(A) \triangleq \min\{d_{\mathrm{E}}(\mathbf{a^{(i)}}, \mathbf{a^{(j)}})\colon\ i, j \in \{1, \dots, T\},\ i \neq j\}.$$

The space $\mathbb{R}^n$ can be considered as a linear space over the field of real numbers $\mathbb{R}$ in a natural way. For every subset $C = \{\mathbf{x^{(i)}} = (x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)}) \mid i = 1, 2, \dots T;\}$ of $\mathbb{R}^n$, denote by C_m^* the multi-set of all sums of at most m different vectors from C. It can happen that C_m^* contains some vectors more than once. In such a case, we obviously have $d_{\mathrm{E}}(C_m^*) = 0$.

A special subset of $\mathbb{R}^n$ is the *unit sphere* Ω_n which consists of all vectors x of norm 1, i.e.

$$\Omega_n \triangleq \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\|_{\mathrm{E}} = 1\}.$$

Any subset of Ω_n is called *spherical code*. Every spherical code is characterized by its *minimum distance*, which is the smallest distance between its different points.

Definition 1. The set C is called an (n, T, d)-spherical code if $C \subset \Omega_n$, $|C| = T$ and $d_{\mathrm{E}}(C) = d$. The parameters n, T and d are called *dimension*, *cardinality* and *minimum distance* of the code C, respectively.

Spherical codes are extensively studied in the literature [12,4,6,10,11,15,2]. The main problem which is considered is finding the largest cardinality of a spherical code with prescribed dimension and minimum distance. Here we impose some stronger conditions on the codes.

Consider the following situation. Suppose that T users use a single channel and the ith user is assigned two codewords, the all-zero vector $\mathbf{0} = (0, 0, \dots, 0) \in \mathbb{R}^n$ and $\mathbf{x^{(i)}} = (x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)}) \in \mathbb{R}^n$. It can be assumed that all non-zero codewords have a unit energy. Before transmitting over the channel the codewords of the users are added as in $\mathbb{R}^n$. Suppose that no more than m users send a non-zero codeword and the channel is disturbed by an additive white Gaussian noise. At the receiver end the decoder has to decide which users have been active, i.e. have sent a non-zero codeword. This model gives rise to the following definition.

Definition 2. The set C is called an (n, d, m, T)-spherical superimposed code (SSC) if $C \subset \Omega_n$, $|C| = T$ and $d_{\mathrm{E}}(C_m^*) = d$. The parameters n, d, m and T are called *dimension*, *minimum distance*, *order* and *cardinality* of C, respectively.

Sometimes the condition that all code vectors have unit norm can be dropped. Instead of this we require that all code points have to be within the unit sphere Ω_n. This simplification is also suggested from the model given above. As we shall see in most of the constructions of SSCs, the exact determination of the minimum distance is impossible. A lower bound on d is computed instead. Therefore, we shall say that any (n, d, m, T)-SSC is also an (n, d_0, m, T)-SSC, where $0 \leqslant d_0 \leqslant d$.

Since we want to include as many users in our system, we are interested in (n, d, m, T)-SSC with as high T as possible for given set of parameters (n, d, m). We denote by

$T(n,d,m)$ the maximal T for which an (n,d,m,T)-SSC exists. The problem of determining the function $T(n,d,m)$ is a very complicated one. Its exact value is known only for a few sets of parameters.

2. Asymptotic results

Before proceeding with the constructions, we give some results on the asymptotic behavior of $T(n,d,m)$ as the dimension tends to infinity. We define the exponent of increase as

$$E(m,d) \triangleq \limsup_{n\to\infty} \frac{1}{n} \log T(n,m,d).$$

The quantity $E(m,d)$ is also known as the best possible *rate* of SSC with parameters m and d.

A natural upper bound on $T(n,d,m)$ is the sphere-packing bound [7,8] which states

$$|C_m^*| = \sum_{i=0}^{m} \binom{T(n,d,m)}{i} \leqslant \left(\frac{m+d/2}{d/2}\right)^n = \left(1+\frac{2m}{d}\right)^n. \quad (1)$$

Direct consequence of this inequality is the following limitation on the exponent

$$E(m,d) \leqslant E_{\mathrm{SP}}(m,d) \triangleq \frac{1}{m} \log m[1+\mathrm{o}(1)], \quad m \to \infty, \quad (2)$$

where $\mathrm{o}(1)$ is a function which tends to zero as m tends to infinity.

A recent significant improvement of (2) is described in [13]. The authors show that almost all sums of exactly m points of a code C are essentially gathered in a ball of radius $\sqrt{m}$ around a certain point. This leads to the following result.

Theorem 3 (Füredi and Ruszinkó [13, Theorem 3.2]). *The following bound on the rate of* SSC*s is valid*

$$E(m,d) \leqslant \frac{1}{2m} \log m[1+\mathrm{o}(1)], \quad m \to \infty. \quad (3)$$

Unfortunately, this bound does not have its match for upper bounding the function $T(n,d,m)$. Theorem 3 shows that the sphere packing bound (1) is actually very weak.

Lower estimations on the rate of SSC are obtained by constructions. As usual in the field of coding theory, the best-known asymptotic lower bound is given by random code construction. We do not discuss any details of the construction itself and give only the final result.

Theorem 4 (Ericson and Györfi [8, Theorem 1]). *For the best possible rate of* SSC*s we have*

$$E(m,d) \geqslant E_{\mathrm{RC}}(m,d) \triangleq \frac{1}{4m} \log m[1+\mathrm{o}(1)], \quad m \to \infty. \quad (4)$$

It seems that the asymptotic behavior of $E(m,d)$ does not depend on the choice of minimum distance d. However, there is still a significant gap between the lower and upper bounds. A very interesting open question is the exact determination of this behavior.

3. Summary of known constructions

Here we describe in brief some known techniques for constructing SSCs. All these constructions can be found in [1].

Construction 5. An orthonormal basis of $\mathbb{R}^n$ forms an $(n,1,n,n)$-SSC.

The points that form an orthonormal basis of $\mathbb{R}^n$ lie on a hyperplane of dimension $n-1$. By projecting them onto $\mathbb{R}^n$ and some rescaling we obtain the so-called simplex code on Ω_{n-1}. It can be easily checked that the following is true.

Construction 6. A simplex code on Ω_{n-1} forms an $(n-1,1,n-1,n)$-SSC.

Before proceeding with more advanced constructions, we give one way to obtain SSCs in two dimensions.

Construction 7. A regular n-gon on Ω_2 forms an $(2,d,k-1,n)$-SSC, where $d>0$ and k is the least non-unit divisor of n.

The drawback of Construction 7 is that the actual minimum distance is difficult to compute.

All methods for deriving SCCs, which we shall give in the end of this section are based on certain mappings from the set $\{0,1,\ldots,p-1\}$ into $\mathbb{R}$ or $\mathbb{R}^2$. They can be defined as follows:

$$\textbf{AM1}: f_1^{(n)}(t)=\frac{t}{\sqrt{n}(p-1)},$$

$$\textbf{AM2}: f_2^{(n)}(t)=\frac{1}{\sqrt{n}}\left(1-\frac{2t}{p-1}\right),$$

$$\textbf{PhM}: f_3^{(n)}(t)=\frac{1}{\sqrt{n}}\left(\cos\frac{2\pi t}{p},\sin\frac{2\pi t}{p}\right).$$

The first more advanced construction of SSCs is described in [7].

Construction 8 (The EG construction). Let C_{b} be a binary linear $[N,K,D]$-code which contains the all-one word. Let $\overline{C_{\mathrm{b}}}$ be the set that is obtained from C_{b} by deleting all words starting with 1 and deleting the first coordinate from the rest. Suppose that

$$D \geqslant d_0 \triangleq \frac{a^2(n+2) - 2a(n+1) + d^2 n}{2a(a-1)} \tag{5}$$

for some d, $0 < d \leqslant 1$, where $a \triangleq \min\{T, 2m\}$ and $n = N - 1$. Applying the mapping **AM2** to $\overline{C_{\mathrm{b}}}$, we obtain (n, d, m, T)-SSC.

As we shall see later on, this construction can be somehow generalized. Another approach is to use p-ary representations of so-called A_s- and B_s-sets. We skip the details given in [1, Chapter 5] and state the result.

Construction 9 (The A construction). Given a primitive polynomial of degree $m+1$ over GF(q), we can obtain (n, d, m, T)-SSC with $T = q+1$ and the following parameters, for any integer $r \geqslant 2$ and for $v = (q^{m+1} - 1)/(q-1)$

$$\mathbf{AM1}: n = \lfloor \log_r v \rfloor + 1, \quad d = \frac{1}{\sqrt{n}(r-1)},$$

$$\mathbf{AM2}: n = \lfloor \log_r v \rfloor + 2, \quad d = \frac{2}{\sqrt{n}(r-1)},$$

$$\mathbf{PhM}: n = 2\lfloor \log_3 v \rfloor + 3, \quad d = \sqrt{\frac{6}{n+1}}, \ r = 3.$$

Another type of superimposed codes considered in [9] can be used for designing SSCs. One possibility of doing this is summarized here.

Construction 10. Let p be a prime number and C_p be an $(N, D, m, T)_p$-superimposed code with $m < p$. Applying the corresponding mapping and appending D zeroes to every codeword of C_p in the last two cases, we obtain (n, d, m, T)-SSC with the following parameters:

$$\mathbf{AM1}: n = N, \quad d \geqslant \frac{1}{p-1}\sqrt{\frac{D}{N}},$$

$$\mathbf{AM2}: n = N + D, \quad d \geqslant \frac{2}{p-1}\sqrt{\frac{D}{N+D}},$$

$$\mathbf{PhM}: n = 2N + D, \quad d \geqslant d_a\sqrt{\frac{D}{N+D}},$$

where d_a is the smallest distance between sums of m vertices of regular p-gon on Ω_2.

Extensive tables with codes obtained from the above constructions can be found in [1]. We shall refer to those tables when we analyze the parameters of the codes constructed in this paper.

4. Codes of dimension two

We pay a special attention to superimposed codes on the unit circle. The reason for doing this is twofold. For the first, it is only Construction 7 that give good codes with $n=2$ and it does not produce codes with even cardinalities. The second goal is to show some exact values of the function $d(n,m,T)$ which is defined as the maximal possible minimum distance of an SSC of dimension n, order m and cardinality T. Codes with parameters $(n,d(n,m,T),m,T)$ will be called d_m-optimal. According to our definition, all $(n,1,m,T)$-SSCs are d_m-optimal since $d \leqslant 1$ is always satisfied. We refer to these cases as trivial. The index m is not redundant. Simple examples are the simplex codes of dimension n which are d_n-optimal but not d_{n+1}-optimal.

In order to simplify the descriptions we introduce some notations. First, we identify $\mathbb{R}^2$ with the set of complex numbers $\mathbb{C}$. Every point $(a,b)\in\mathbb{R}^2$ is associated with the number $a+\mathrm{i}b=\rho\mathrm{e}^{\mathrm{i}\varphi}$, where $\mathrm{i}^2=-1$. Every set on the unit circle can be represented by a set of angles $\varphi\in[0,2\pi)$ corresponding to its points. For example the set $\mathscr{C}_k \triangleq \{\varphi_j=2j\pi/k,\ j=0,1,\dots,k-1\}$ represents a regular k-gon which has vertex (1,0). In fact, all the codes that will be given here can be described as a subsets of $\mathscr{C}_k$ for some $k\in\mathbb{N}$.

A natural way of obtaining codes with even cardinalities is to take away one point from the regular polygon with one more vertex. However, the following construction gives better minimum distances.

Construction 11. Let T be an even number which is not a power of 2. Let p be the smallest odd prime divisor of T. Choose the set $\mathscr{B}_p^T$ to be the subset of $\mathscr{C}_{2T}$ consisting of the angles

$$\varphi_k^i=\left(\frac{2k}{p}+\frac{i}{T}\right)\pi,\quad k=0,1,\dots,p-1,\quad i=0,1,\dots,T/p-1.$$

The exact determination of the minimum distance of the codes $\mathscr{B}_p^T$ is not known in the general case. We have computed it for the case $p=3$.

Theorem 12. *Let T be a positive integer number divisible by* 6. *Then the codes $\mathscr{B}_3^T$ given in Construction* 11 *have parameters $(n,d,m,T)=(2,4\sin\pi/T\sin\pi/2T,2,T)$.*

Proof. First, we observe that the set $(\mathscr{B}_3^T)_2^*$ is preserved by the rotations through angle $2\pi/3$ and center in the origin. It is also kept by the reflections in the lines along the vectors corresponding to the angles $((2i+1)T-3)/6T\pi$, $i=0,1,2$. Thus we can consider the non-zero points of $(\mathscr{B}_3^T)_2^*$ which correspond to angles in the interval $[0,2\pi/3]$. These points can be divided in three sets defined as $B_1=\{\varphi_0^i\mid i=0,1,\dots,T/3-1\}$, $B_2=\{\varphi_0^i+\varphi_0^j\mid i,j=0,1,\dots,T/3-1,\ i\neq j\}$ and $B_3=\{\varphi_0^i+\varphi_1^j\mid i,j=0,1,\dots,T/3-1\}$. It is easy to see that the distance between two points from different sets as well as the distance of every point to the origin is at least $2\sin\pi/2T$, which is the side-length of the regular $2T$-gon. Further, the points of B_3 can be divided in "levels" by their Euclidean norm. The minimum distance between the different levels is $2\sin\pi/2T$ and between the points on the level of radius r is $2r\sin\pi/2T$. The innermost level with at least 2 points has $r = 2\sin\pi/T$ and thus $d_{\mathrm{E}}(B_3) = 4\sin\pi/T\sin\pi/2T$. By similar arguments, we can deduce $d_{\mathrm{E}}(B_2) = 4\sin\pi/T\sin\pi/2T$. Clearly, $d_{\mathrm{E}}(B_1) = 2\sin\pi/2T$ which concludes our proof. □

For the case $p > 3$, we claim that the minimum distance of the constructed codes is non-zero. Before proceeding with the proof of this fact, we need the following lemma.

Lemma 13. *Let T be an even positive number that is not a power of 2 and p be its least odd prime divisor. Then there are no opposite vectors in $\mathscr{B}_p^T$, i.e. vectors with zero sum. Moreover, all regular p-gons with vertices in $\mathscr{C}_{2T}$ are either completely included or does not have points in $\mathscr{B}_p^T$.*

Proof. Suppose first that there are opposite points in $\mathscr{B}_p^T$. Then

$$\pi = |\varphi_k^i - \varphi_l^j| = \left|\frac{2(k-l)}{p} - \frac{i-j}{T}\right|\pi$$

for some integers i,j,k,l such that $i,j \in [0, T/p-1]$ and $k,l \in [0, p-1]$. This is impossible since p is an odd number and $|i-j|/T < 1/p$. The second part follows directly from the easy observation that all regular p-gons, which are subsets of $\mathscr{C}_{2T}$ are $\{\varphi_k^i\}_{k=0}^{p-1}$ for $i = 0, 1, \ldots, 2T/p - 1$. □

Now we can give the main result concerning Construction 11.

Theorem 14. *The codes $\mathscr{B}_p^T$ described in Construction 11 are $(2, d, p-1, T)$-SSCs where $d > 0$.*

Proof. Suppose that $d = 0$, which means that we have two different sets M and N of up to $p-1$ points in $\mathscr{B}_p^T$ which have the same sum. We can assume that $M \cap N = \emptyset$. Let us denote by $\bar{N}$ the set of opposite vectors to those in N. Then the sum of the vectors in $M \cup \bar{N}$ is the zero vector. Since $M \cup \bar{N} \subseteq \mathscr{C}_{2T}$ this can happen only if the points in $M \cup \bar{N}$ are all the vertices of a regular l-gon, where $l|T$ and $l \geqslant 2$. We have $1 \leqslant |M \cup \bar{N}| \leqslant 2(p-1)$ and from the definition of T and p we get two possible cases, namely $|M \cup \bar{N}|$ even or $|M \cup \bar{N}| = p$. Both cases are excluded by Lemma 13.

Since the angle between any two lines through the origin and the points of a $(2, d, m, T)$-SSC with $m \geqslant 2$ must be at least $2\arcsin(d/2)$, we obtain the following upper bound on the minimum distance of such a code. □

Proposition 15. *If there exists a $(2, d, m, T)$-SSC with $m \geqslant 2$ and $T \geqslant 3$, then $d \leqslant 2\sin(\pi/(2T))$.*

Proof. The only thing we must see is the obvious fact that the minimum angle between T lines through the origin in $\mathbb{R}^2$ is at most π/T. □

Table 1
Comparison of constructions 7 and 11 for codes of dimension $n=2$ and order $m=2$

T	d_7	d_{11}	$d_{\mathbf{ub}}$
6	0.24697960	0.51763809	0.51763809
10	0.16037889	0.17557050	0.31286893
12	0.11538526	0.13513066	0.26105238
14	0.08693075	0.09965775	0.22392895
18	0.05436845	0.06053774	0.17431149
20	0.04455177	0.04909482	0.15691819
22	0.03716936	0.04061049	0.14267837
24	0.03147895	0.03414728	0.13080626
26	0.02700081	0.02911129	0.12075699
28	0.02341378	0.02511159	0.11214089
30	0.02049636	0.02188238	0.10467191
34	0.01608661	0.01704508	0.09236691
36	0.01439706	0.01520672	0.08723877
38	0.01296024	0.01365035	0.08264994
40	0.01172818	0.01232115	0.07851963

For the special case of $m=2$, the bound from Proposition 15 is asymptotically better than the sphere packing bound given in (1) as $T\to\infty$. It is not surprising that for larger m we have the opposite situation. A natural explanation is that the limitation on the angles of the lines is quite weak in those cases.

To see the advantages of Construction 11 compared with Construction 7, we have computed the actual minimum distances d for order $m=2$ and even cardinalities up to 40. The results are given in Table 1. The notation d_i refers to the minimum distance of the codes obtained from the corresponding construction. The codes from Construction 7 are obtained by removing one point from the vertices of a regular $(T+1)$-gon. We list the corresponding upper bound obtained by Proposition 15 in the last column of the table.

Another possibilities for choosing some points of $\mathscr{C}_k$ to obtain $(2,d,m,T)$-SSCs can be investigated. This idea is promising as we can see from the following example.

Example 16. The code $\mathscr{C}_{10}^{0,1,4,7}$ consisting of vectors corresponding to the angles $0,\pi/5$, $4\pi/5$ and $7\pi/5$, which is a subset of $\mathscr{C}_{10}$ is a $(2,2\sin(\pi/10),2,4)$-SSC.

It is possible to show that the code in Example 16 satisfies $d(2,2,4)=2\sin(\pi/10)$. With the aid of the bound from Proposition 15 we are able to determine two more values of the function $d(n,m,T)$, namely $d(2,2,3)=1$ and $d(2,2,6)=2\sin(\pi/12)$. The codes achieving these values are $\mathscr{C}_3$ and $\mathscr{B}_3^6$, respectively. Observe that $\mathscr{C}_3$ is d_2-optimal, but clearly not d_3-optimal. Further geometrical reasons reveal that $d(2,2,5)=d(2,2,6)=2\sin(\pi/12)$. The known cases of d_m-optimal codes with $d<1$ are summarized in Table 2.

Table 2
Known d_m-optimal spherical superimposed codes with $d < 1$

n	m	T	$d(n,m,T)$
2	2	4	$2\sin(\pi/10) \approx 0.61802399$
2	2	5	$2\sin(\pi/12) \approx 0.51763809$
2	2	6	$2\sin(\pi/12) \approx 0.51763809$

5. Non-optimality of the simplex codes

Let us denote by $\mathscr{S}_n$ the simplex code on Ω_n. A possible way of obtaining $\mathscr{S}_n$ is the following. Take the standard basis $\mathbf{e}^{(i)}$, $i=1,\dots,n+1$ of the space $\mathbb{R}^{n+1}$, consisting of vectors with one in the ith position and zeros elsewhere. These vectors lie on the intersection of Ω_{n+1} and the hyperplane in $\mathbb{R}^{n+1}$ defined by the equation $\sum_{i=1}^{n+1} x_i = 1$. Define $\mathbf{f}^{(i)} = \sqrt{(n+1)/n}(\mathbf{e}^{(i)} - \mathbf{e}/(n+1))$, $i=1,\dots,n+1$, where $\mathbf{e}=(1,1,\dots,1)$. The points $\mathbf{f}^{(i)}$ lie simultaneously on hyperplane through the origin in $\mathbb{R}^{n+1}$ and Ω_{n+1}. Thus they can be considered as points on Ω_n. We compute the inner product between the different vectors to be

$$\langle \mathbf{f}^{(i)}, \mathbf{f}^{(j)} \rangle = \frac{n+1}{n}\left(\langle \mathbf{e}^{(i)}, \mathbf{e}^{(j)} \rangle - \frac{\langle \mathbf{e}^{(i)} + \mathbf{e}^{(j)}, \mathbf{e} \rangle}{n+1} + \frac{\langle \mathbf{e}, \mathbf{e} \rangle}{(n+1)^2} \right) = -\frac{1}{n}.$$

The simplex codes are optimal as spherical codes, i.e. they possess the best possible minimum distance among all codes of dimension n and cardinality $n+1$. They also represent sets of least cardinality of points for interpolation formula for computing integrals on Ω_n which are exact for every n-variable polynomial of total degree 2 or lower. As we mentioned in the previous section, the codes $\mathscr{S}_n$ are d_n-optimal but not d_{n+1}-optimal since they are $(n,0,n+1,n+1)$-SSCs. The question we consider here is whether the simplex codes have the best cardinality among all SSCs with parameters $(n,d,m)=(n,1,n)$, i.e. whether $T(n,1,n)=n+1$ for every integer $n \geqslant 2$. We saw that this is the case when $n=2$. However, this is not true in the general case. To prove this fact we shall show that for $n \geqslant 10$ we can add a point to $\mathscr{S}_n$ so that we preserve its superimposed parameters. We have the following representation of every point on Ω_n.

Lemma 17. *Let $\mathscr{S}_n = \{\mathbf{f}^{(i)}, i=1,\dots,n+1\} \subset \Omega_n$ be the simplex code defined above. Then every point $\mathbf{x}$ on Ω_n can be represented in the form $\mathbf{x} = \sum_{i=1}^{n+1} x_i \mathbf{f}^{(i)}$,* where $\sum_{i=1}^{n+1} x_i = 0$ and $\sum_{i=1}^{n+1} x_i^2 = n/(n+1)$.

Proof. Every point $\mathbf{x} \in \Omega_n$ is an image of some point $\mathbf{y}=(y_1,\dots,y_{n+1}) \in \Omega_{n+1}$, with $\sum_{i=1}^{n+1} y_i = 1$ under the transformation defining the simplex code. This means that the following holds

$$\mathbf{x} = \sqrt{\frac{n+1}{n}}\left(\mathbf{y} - \frac{\mathbf{e}}{n+1}\right) = \sum_{i=1}^{n+1}\left(y_i - \frac{1}{n+1}\right)\sqrt{\frac{n+1}{n}}\left(\mathbf{e}^{(i)} - \frac{\mathbf{e}}{n+1}\right).$$

Define $x_i = y_i - 1/(n+1)$ for $i = 1,\ldots,n+1$. Clearly, $\sum_{i=1}^{n+1} x_i = 0$ and for the sum of the squares we have

$$\sum_{i=1}^{n+1} x_i^2 = \sum_{i=1}^{n+1} y_i^2 - \frac{2}{n+1}\sum_{i=1}^{n+1} y_i + \frac{n}{(n+1)^2} = \frac{n}{n+1}.$$

Throughout this section, we represent a point $\mathbf{x} \in \Omega_n$ as in Lemma 17. It is easy to compute that $\langle \mathbf{x}, \mathbf{f}^{(i)} \rangle = (n+1)/n x_i$. Suppose now that the set $\mathscr{S}_n \cup \mathbf{x}$ is an $(n,1,n,n+2)$-SSC. Without lost of generality, we assume $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{n+1}$. The next statement gives the conditions which the coordinates x_i, $i = 1,\ldots,n+1$ must satisfy. □

Lemma 18. *The set* $\mathscr{S}_n \cup \mathbf{x}$ *is an* $(n,1,n,n+2)$-SSC *if and only if the inequalities*

$$\sum_{i=1}^{k} x_i \geqslant \frac{k^2}{2(n+1)} - \frac{k}{2}, \quad k = 1,\ldots,n \tag{6}$$

are satisfied.

Proof. We first show the sufficiency of conditions (6). Let I and J be subsets of $\{1,2,\ldots,n+1\}$, such that $I \cap J = \emptyset$, $|I| = k \leqslant n-1$ and $|J| = l \leqslant n$. Since $\mathscr{S}_n$ is an $(n,1,n,n+1)$-SSC we only have to prove that $d(\mathbf{x} + \sum_{i\in I} \mathbf{f}^{(i)}, \sum_{j\in J} \mathbf{f}^{(j)}) \geqslant 1$ for arbitrary choice of the sets I and J having the above properties. Since we chose the point $\mathbf{x}$ to have $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{n+1}$ we consequently obtain

$$\begin{aligned}
&d^2\left(x + \sum_{i\in I}\mathbf{f}^{(i)}, \sum_{j\in J}\mathbf{f}^{(j)}\right)\\
&\quad = \left\langle x + \sum_{i\in I}\mathbf{f}^{(i)} - \sum_{j\in J}\mathbf{f}^{(j)}, x + \sum_{i\in I}\mathbf{f}^{(i)} - \sum_{j\in J}\mathbf{f}^{(j)}\right\rangle\\
&\quad = 1 + 2\,\frac{n+1}{n}\left(\sum_{i\in I} x_i - \sum_{j\in J} x_j + \frac{k+l}{2}\right) - \frac{(k-l)^2}{n}\\
&\quad \geqslant 1 + 2\,\frac{n+1}{n}\left(\sum_{i=1}^{k} x_i - \sum_{j=n+2-l}^{n+1} x_j + \frac{k+l}{2}\right) - \frac{(k-l)^2}{n}\\
&\quad \geqslant 1 + 2\,\frac{n+1}{n}\left(\sum_{i=1}^{k} x_i + \sum_{j=1}^{n+1-l} x_j + \frac{k+l}{2}\right) - \frac{(k-l)^2}{n}\\
&\quad \geqslant 1 + \frac{k^2+l^2}{n} - \frac{(k-l)^2}{n} = 1 + \frac{2kl}{n} \geqslant 1.
\end{aligned}$$

The necessity of conditions (6) follows in a similar way. This can be shown by using the inequalities $\|\mathbf{x} + \sum_{i=1}^{k} \mathbf{f}^{(i)}\| \geqslant 1$, $k = 1, \dots, n-1$ and $\|\mathbf{x} + \sum_{j=1}^{n} \mathbf{f}^{(j)}\|^2 = d^2(x, f_{n+1}) \geqslant 1$. □

Now we can prove the fact that it is possible to extend the simplex code $\mathscr{S}_n$ with additional point whenever $n \geqslant 10$.

Theorem 19. *Let $n \geqslant 10$ and define the point $\mathbf{x}=\sum_{i=1}^{n+1} \mathbf{f}^{(i)}$, where $x_i=\sqrt{3/(n+2)}(2i-2-n)/(n+1)$ for $i = 1, 2, \dots, n+1$. Then we have $\mathbf{x} \in \Omega_n$ and the set $\mathscr{S}_n \cup \mathbf{x}$ is an $(n, 1, n, n+2)$-SSC.*

Proof. The definition of x is correct since $(n+1)\sqrt{(n+2)/3}\sum_{i=1}^{n+1} x_i = \sum_{i=1}^{n+1}(2i-2-n) = -(n+1)n - 2(n+1) + 2\sum_{i=1}^{n+1} i = 0$. To show that $x \in \Omega_n$, we compute

$$\sum_{i=1}^{n+1} x_i^2 = \frac{3}{(n+2)(n+1)^2}\sum_{i=0}^{n}(n-2i)^2$$
$$= \frac{3}{(n+2)(n+1)^2}\left[(n+1)n^2 - 4n\,\frac{n(n+1)}{2} + 4\,\frac{n(n+1)(2n+1)}{6}\right]$$
$$= \frac{n}{n+1}.$$

The inequalities $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{n+1}$ are obvious. In order to use Lemma 18 we determine

$$\sum_{i=1}^{k} x_i = \sqrt{\frac{3}{n+2}}\,\frac{1}{n+1}\sum_{i=1}^{k}(2i-2-n) = \sqrt{\frac{12}{n+2}}\left(\frac{k^2}{2(n+1)} - \frac{k}{2}\right).$$

Since the right-hand side of (6) is negative for $k = 1, \dots, n$ it follows that it is satisfied for all $n \geqslant 10$ and thus $\mathscr{S}_n \cup \mathbf{x}$ is an $(n, 1, n, n+2)$-SSC. □

As a consequence of the previous theorem we get the following lower bound on $T(n, 1, n)$.

Corollary 20. *For every $n \geqslant 10$ we have $T(n, 1, n) \geqslant n+2$.*

For the cases $n \leqslant 9$ it can be shown that it is not possible to add a point to the code $\mathscr{S}_n$ while preserving its superimposed properties. However, it does not mean that the simplex code is optimal in these cases. We already proved that $T(2, 1, 2) = 3$ and thus $\mathscr{S}_2$ is optimal. The optimality of $\mathscr{S}_n$ for $n = 3, 4, \dots, 9$ is still an open problem.

6. Generalization of the EG construction

Now we discuss some necessary conditions for a spherical code to posses certain superimposed properties. Our investigations follow the spirit of the ideas behind the

EG construction. If the correlation between the different points of certain spherical code lie in a given interval $[s_1, s_2] \subset [-1, 1)$, under some conditions we can claim that this code has given order m as a superimposed code. These conditions look very complicated, but they prove to be useful in many cases.

We begin with defining a non-negative valued function $g(m, T, s_1, s_2)$ in the following manner. Let $g(m, T, s_1, s_2) = g_i(m, T, s_1, s_2)$ whenever $(m, T, s_1, s_2) \in M_i \subset \mathbb{Z}_+^2 \otimes \mathbb{R}^2$, $i = 1, 2, 3, 4, 5, 6$. The functions $g_i(m, T, s_1, s_2)$ are defined as

$$g_i^2(m, T, s_1, s_2) = \begin{cases} (2m^2 + T^2 - 2mT - T)s_1 + 2m(m - T)s_2 + T & \text{if } i = 1, \\ (T(T-2)s_1 - T^2 s_2 + 2T)/2 & \text{if } i = 2, \\ 2m((m-1)s_1 - ms_2 + 1) & \text{if } i = 3, \\ m((m-1)s_1 + 1) & \text{if } i = 4, \\ 1 & \text{if } i = 5, \\ 0 & \text{if } i = 6. \end{cases}$$

The regions M_i, $i = 1, \ldots, 6$ depend on the choice of parameters m, T, s_1 and s_2. Their description is given below.

$$M_1 = \left\{ (m, T, s_1, s_2) \,|\, -\frac{1}{m-1} \leqslant s_1 \leqslant -\frac{1}{T}, \frac{(T-m-1)s_1 + 1}{2m} \leqslant s_2 \leqslant -s_1, \right.$$
$$\left. \frac{(T-1)(Ts_1+1)}{2m(T-m)} - s_1 \leqslant s_2 \leqslant \frac{T((T-1)s_1+1)}{2m(T-m)} - s_1, m < T < 2m \right\},$$

$$M_2 = \left\{ (m, T, s_1, s_2) \,|\, m \leqslant T < 2m, -\frac{1}{T-1} \leqslant s_1 \leqslant 0, s_2 \geqslant -s_1, \right.$$
$$\left. \frac{T(T-2)s_1 + 2(T-1)}{T^2} \leqslant s_2 \leqslant \frac{(T-2)s_1 + 2}{T} \right\},$$

$$M_3 = \left\{ (m, T, s_1, s_2) \,|\, 4 \leqslant 2m \leqslant T, -\frac{1}{m-1} \leqslant s_1 \leqslant 0, s_2 \geqslant \frac{(m-1)s_1 + 1}{2m}, \right.$$
$$\left. \frac{(m-1)s_1 + 1}{m} - \frac{1}{2m^2} \leqslant s_2 \leqslant \frac{(m-1)s_1 + 1}{m} \right\},$$

$$M_4 = \left\{ (m, T, s_1, s_2) \,|\, 2 \leqslant m \leqslant T, -\frac{1}{m-1} \leqslant s_1 \leqslant -\frac{1}{m}, s_2 = s_1 \text{ if } T = m, \right.$$
$$\left. s_2 \leqslant \frac{(a-m-1)s_1 + 1}{2m} \text{ if } m \neq T \text{ and } a = \min\{T, 2m\}, s_2 \geqslant s_1 \right\},$$

$$M_5 = \left\{ (m, T, s_1, s_2) \,|\, 2 \leqslant m \leqslant T, s_2 \leqslant \left(1 - \frac{2}{T}\right) s_1 + 2\,\frac{(T-1)}{T^2} \text{ if } m \leqslant T < 2m, \right.$$

$$s_2 \leqslant \frac{(a-1)(as_1+1)}{2m(a-m)} - s_1 \text{ if } m \neq T \text{ and } a = \min\{T, 2m\},$$

$$\left. s_2 \geqslant s_1, -\frac{1}{m} \leqslant s_1 \leqslant 0 \right\},$$

$$M_6 = \mathbb{Z}_+^2 \otimes \mathbb{R}^2 \Bigg\backslash \bigcup_{i=1}^{5} M_i$$

The function $g(m, T, s_1, s_2)$ gives actually a lower bound on the minimum distance of a certain spherical code considered as an SSC. The result is given in the next statement.

Theorem 21. *Let $m \geqslant 2$ be an integer and C be an (n, T, d_0)-spherical code such that $\langle \mathbf{x}, \mathbf{y} \rangle \in [s_1, s_2]$ for every $\mathbf{x} \neq \mathbf{y}$ in C. Then C is a $(n, d = g(m, T, s_1, s_2), m, T)$-SSC.*

The proof of Theorem 21 can be found in [5]. Two particular cases of special interest need to be stated here. The first is when the interval $[s_1, s_2]$ is symmetric around zero.

Corollary 22. *Let C be an (n, T, d_0)-spherical code with inner products within the interval $[s, -s]$, where $-1 \leqslant s \leqslant 0$. Let m be a positive integer and define $a = \min\{2m, T\}$. Then C is an $(n, 1, m, T)$-SSC if $s \in [-1/a, 0]$ and $(n, \sqrt{a(1+(a-1)s)}, m, T)$-SSC if $s \in [-1/(a-1), -1/a]$.*

Another interesting case is when we obtain superimposed codes with $d = 1$ and many points.

Corollary 23. *Let C be a (n, T, d_0)-spherical code with inner products in the interval $[s_1, s_2]$ and let $m \leqslant T/2$ be an integer number. If $s_1 \in [-1/m, 0]$ and*

$$s_2 \in \left[s_1, \frac{((m-1)s_1+1)}{m} - \frac{1}{2m^2} \right]$$

then C is an $(n, 1, m, T)$-SSC.

We can use Corollary 22 to see that Construction 8 is a particular case of Theorem 21. The only thing to show is that the mapping **AM2** applied on the binary code described in Construction 8 results in a spherical code with inner products in the interval $[(2D-n-2)/n, -(2D-n-2)/n]$. Indeed, if the Hamming distance between two binary vectors of length n is w, then the inner product between their images on Ω_n after applying the mapping **AM2** will be $1 - 2w/n$. It remains to see that the non-zero Hamming distances of the pre-image lie in the interval $[D, n+1-D]$. Condition (5) can be eased a little since the inner products actually lie in a smaller interval $[(2D-n-2)/n, (n-2D)/n]$.

Some applications of Theorem 21 will be given until the end of the section. First, we consider spherical codes obtained from codes in the Grassmannian space $\mathbf{G}(n, 1)$

Table 3
Spherical superimposed codes derived from the best codes in $\mathbf{G}(n,1)$ known

n	d	m	T	n	d	m	T	n	$d \geqslant$	m	T	n	$d \geqslant$	m	T
6	1	2	8	11	1	3	14	6	0.453	2	12	12	0.930	3	17
7	1	2	10	12	1	3	16	7	0.819	2	14	12	0.342	3	20
8	1	2	13	13	1	3	19	9	0.442	2	30	13	0.831	3	20
9	1	2	18	14	1	3	20	10	0.554	2	40	14	0.993	3	21
10	1	2	20	15	1	3	23	11	0.289	2	54	14	0.808	3	23
11	1	2	26	16	1	3	26	11	0.161	2	60	15	0.985	3	24
12	1	2	39	12	1	4	14	12	0.826	2	48	15	0.227	3	30
13	1	2	52	13	1	4	15	13	0.808	2	54	16	0.600	3	31
14	1	2	54	14	1	4	16	14	0.999	2	55	11	0.663	4	13
15	1	2	50	15	1	4	18	8	0.719	3	10	13	0.663	4	16
16	1	2	50	16	1	4	20	9	0.719	3	12	14	0.947	4	17
9	1	3	11	15	1	5	17	10	0.530	3	14	15	0.776	4	19
10	1	3	12	16	1	5	18	11	0.483	3	17	14	0.648	5	16

(also known as real projective space $\mathbf{P}\mathbb{R}^{n-1}$). This space consists of all lines in $\mathbb{R}^n$ passing through the origin. A spherical code can be obtained from a set of such lines (called code in $\mathbf{G}(n,1)$) in an obvious way. If L is a code in $\mathbf{G}(n,1)$ of cardinality $|L| = M$, then by choosing the intersection points of the lines from L with the unit sphere we obtain spherical code C with $2M$ points. Such a code is called antipodal. This code cannot be used as a superimposed code since the sum of the two points obtained from one line is the all-zero vector. If we choose only one of the points for each line in L in an arbitrary way, we get a spherical code C with M points. The code L is characterized by the minimum angle α between its lines. An obvious statement is that for any two points $\mathbf{x}, \mathbf{y} \in C$ we have $|\langle \mathbf{x}, \mathbf{y} \rangle| \leqslant \cos\alpha$.

Constructions of codes in $\mathbf{G}(n,1)$ can be found in [3]. The parameters of the best codes known can be retrieved from [16]. Table 3 gives the parameters of some SSCs obtained from these codes with the help of the construction given above. The lower bound on the minimum distance is computed with the help of Corollary 22.

A general construction of sets of equiangular lines is given in [17, Theorem 6.3]. It turns out that these sets are optimal codes in $\mathbf{G}(n,1)$ [3]. We can use them to obtain good SSCs.

Proposition 24. *Let n be an odd positive integer number such that conference matrix of order $2n$ exists. If m is a positive integer number for which $n \geqslant 2m^2 + 1$, then there exists $(n, 1, m, 2n)$-SSC.*

Another approach is a direct use of spherical codes and check their superposition properties with the help of Theorem 21. A table of optimal spherical codes is given in [15, Table 9.1]. We obtain two series of SSCs.

Proposition 25. *If n is a positive integer number and $q \geqslant 3$ is a power of a prime then there exist $(n, 1, n, n+1)$- and $(q(q^2 - q + 1), 1, q - 1, (q+1)(q^3 + 1))$-SSCs.*

The first sequence is obtained from simplex codes and was already mentioned. The second corresponds to a construction of optimal spherical codes described in [14]. From the other codes in [15, Table 9.1] we obtain only three other interesting SSCs. They have parameters $(21,1,2,162)$, $(22,1,2,100)$ and $(22,1,2,275)$, respectively.

7. Tensor product construction

Here we describe one way of combining two SSCs into another and discuss its parameters. The codes that we obtain are not asymptotically "good", but they represent the best codes known for certain small parameters.

Let us consider the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^k$ for some positive integer numbers n and k. Denote the standard bases of these spaces by $\mathbf{e}^{(i)}$, $i=1,\dots,n$ and $\mathbf{f}^{(j)}$, $j=1,\dots,k$, respectively. We consider the linear mapping

$$\otimes : \mathbb{R}^n \times \mathbb{R}^k \mapsto \mathbb{R}^{nk}$$

defined on the basis as

$$\otimes(\mathbf{e}^{(i)}, \mathbf{f}^{(j)}) = \mathbf{h}_{ij}$$

and extended on the whole set $\mathbb{R}^n \times \mathbb{R}^k$ by linearity. Here $\{\mathbf{h}_{ij}\}_{i=1,j=1}^{n,k}$ denotes the standard basis of $\mathbb{R}^{nk}$. We shall write $x \otimes y$ instead of $\otimes(x,y)$. This means that for every two vectors $\mathbf{x} = \sum_{i=1}^n x_i \mathbf{e}^{(i)} \in \mathbb{R}^n$ and $\mathbf{y} = \sum_{j=1}^k y_j \mathbf{f}^{(j)} \in \mathbb{R}^k$ we have

$$x \otimes y = \sum_{i=1}^n \sum_{j=1}^k x_i y_j.\mathbf{e}^{(i)} \otimes \mathbf{f}^{(j)} = \sum_{i=1}^n \sum_{j=1}^k x_i y_j.\mathbf{h}_{ij}.$$

This mapping is known as tensor product of $\mathbb{R}^n$ and $\mathbb{R}^k$. We give some properties of the tensor product. Let $\mathbf{x}, \mathbf{z} \in \mathbb{R}^n$ and $\mathbf{y}, \mathbf{w} \in \mathbb{R}^k$. Direct consequence of the definitions is the following equality:

$$\langle \mathbf{x} \otimes \mathbf{y}, \mathbf{z} \otimes \mathbf{w} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle \langle \mathbf{y}, \mathbf{w} \rangle. \tag{7}$$

Particular case is the property $\Omega_n \otimes \Omega_k \subseteq \Omega_{nk}$. Thus the tensor product of two spherical codes is also a spherical code. Moreover, the inner products of the obtained code can be determined with the help of the inner products of the building codes. This is stated below.

Proposition 26. *Let $C_i \subset \Omega_{n_i}$, $i=1,2$ be two spherical codes and let $A_i = \{s_1^{(i)}, s_2^{(i)}, \dots, s_{k_i-1}^{(i)}, s_{k_i}^{(i)} = 1\}$, $i=1,2$ be the sets of their inner products. Then the set of the inner products of the code $C = C_1 \otimes C_2$ is $A = \{s_{l_1}^{(1)} s_{l_2}^{(2)} \mid l_i = 1,2,\dots k_i,\ i=1,2\}$.*

We can use Theorem 21 to find the parameters of $C = C_1 \otimes C_2$ as a superimposed code. In order to do this we have to find the interval where the non-unit inner products of C are situated. We can find it with the help of Proposition 26. Namely, if $[s_1^{(i)}, s_{k_i-1}^{(i)}]$, $i=1,2$ are the corresponding intervals for the codes C_i, $i=1,2$, then the non-unit inner products of $C = C_1 \otimes C_2$ are within the interval

$$[\min\{s_1^{(1)}, s_1^{(2)}, s_{k_1-1}^{(1)} s_{k_2-1}^{(2)}\}, \max\{s_{k_1-1}^{(1)}, s_{k_2-1}^{(2)}, s_1^{(1)} s_1^{(2)}\}].$$

We cannot expect the tensor product code to have higher order than any of its building codes. Indeed, due to (7) an isometric image, $\mathbf{x}^{(1)} \otimes C_2$ or $C_1 \otimes \mathbf{x}^{(2)}$, where $\mathbf{x}^{(i)} \in C_i$, $i = 1,2$, of any of the codes C_i, $i = 1,2$ is included as a subcode in $C = C_1 \otimes C_2$.

The first application that we show is combining orthonormal basis from Construction 5 and simplex code $\mathscr{S}_n$.

Theorem 27 (Tensor product of orthogonal and simplex code). *There exists an* $(n_1 n_2, 1, n_2, n_1(n_2+1))$-SSC *for every* $n_1, n_2 \in \mathbb{N}$.

Proof. If we take the tensor product of an orthonormal basis of $\mathbb{R}^{n_1}$ and a simplex code $\mathscr{S}_{n_2} \subset \mathbb{R}^{n_2}$ we obtain a spherical code of dimension $n_1 n_2$ with inner products in the set $\{-1/n_2, 0, 1\}$. We can apply Corollary 23 with $s_1 = -1/n_2$ and $s_2 = 0$ to see that it is actually an $(n_1 n_2, 1, n_2, n_1(n_2+1))$-SSC.

It is natural to consider the tensor product of spherical codes with small number of inner products between their points. The next step is to investigate the parameters of SCCs of type $\mathscr{S}_{n_1} \otimes \mathscr{S}_{n_2}$. We can assume that $n_1 \leqslant n_2$. Since the inner products of these codes are in the interval $[-1/n_1, 1/(n_1 n_2)]$, Corollary 23 gives the following result.

Lemma 28. *For every* $n_1, n_2 \in \mathbb{N}$ *such that* $n_2 \geqslant 2n_1$ *there exists an* SSC*s with parameters* $(n,d,m,T) = (n_1 n_2, 1, n_1, (n_1+1)(n_2+1))$.

It is easy to see that the order of $\mathscr{S}_{n_1} \otimes \mathscr{S}_{n_1}$ is less than n_1 since there are two equal sums of the vectors from two different sets of cardinality n_1. If the tensor product consists of the points $\{\mathbf{a_i} \otimes \mathbf{b_j} \mid i,j = 1,2,\ldots,n_1+1\}$ then these sets can be defined as $\{\mathbf{a_i} \otimes \mathbf{b_{n_1+1}} \mid i = 1,\ldots,n_1\}$ and $\{\mathbf{a_{n_1+1}} \otimes \mathbf{b_j} \mid j = 1,\ldots,n_1\}$. It is possible to prove that $\mathscr{S}_{n_1} \otimes \mathscr{S}_{n_1}$ is an $(n_1^2, 1, n_1-1, (n_1+1)^2)$-SSC.

The cases not covered by Lemma 28 are given below.

Theorem 29 (Tensor product of simplex codes). *Let* $n_1, n_2 \in \mathbb{N}$ *and* $n_1 \leqslant n_2$. *Then there exists an* $(n_1 n_2, 1, n_1 - \delta_{n_1 n_2}, (n_1+1)(n_2+1))$-SSC, *where* δ *is the Kroneker symbol.*

The above codes are with best known cardinality for the corresponding parameters at least in the cases $(n_1, n_2) \in \{(2,3),(3,4),(3,5),(3,6),(3,7),(3,8),(4,5),(4,6),(4,7),(4,8),(5,6),(5,7),(5,8)\}$. For large $n_2 - n_1$ this construction does not reveal codes with "good" cardinality, but the regular structure can somehow facilitate the decoding procedure.

References

[1] P.-O. Anderson, Superimposed codes for the Euclidean channel, Ph.D. Thesis no. 342, Linköping University, Linköping, Sweden, 1994.

[2] P.G. Boyvalenkov, D.P. Danev, S.P. Bumova, Upper bounds on the minimum distance of spherical codes, IEEE Trans. Inform. Theory 42 (5) (1996) 1576–1581.
[3] J.H. Conway, R.H. Hardin, N.J.A. Sloane, Packing lines, planes, etc., packings in grassmannian spaces, Experimental Math. 5 (1996) 139–159.
[4] J.H. Conway, N.J.A. Sloane, Sphere Packings, Lattices and Groups, 3rd Edition, Springer, New York, 1999 (with additional contributions by E. Bannai, R.E. Borcherds, J. Leech, S.P. Norton, A.M. Odlyzko, R.A. Parker, L. Queen, B.B. Venkov).
[5] D. Danev, Superimposed codes in $\mathbb{R}^n$ obtained from spherical codes, in: Proceedings of the International Workshop on Coding and Cryptography, Paris, France, January 2001, pp. 157–166.
[6] S.M. Dodunekov, T. Ericson, V.A. Zinovév, A concatenation method for constructing spherical codes in n-dimensional Euclidean space, Problemy Peredachi Informatsii 27 (4) (1991) 34–38.
[7] T. Ericson, L. Györfi, Superimposed codes in $\mathbb{R}^n$, Internal Report LiTH-ISY-I-0818, Department of Electrical Engineering, Linköping University, Linköping, Sweden, October 1986.
[8] T. Ericson, L. Györfi, Superimposed codes in $\mathbb{R}^n$, IEEE Trans. Inform. Theory 34 (4) (1998) 877–880.
[9] T. Ericson, V.I. Levenshtein, Superimposed codes in the hamming space, IEEE Trans. Inform. Theory 40 (6) (1994) 1882–1893.
[10] T. Ericson, V. Zinoviev, On spherical codes generating the kissing number in dimensions 8 and 24, Discrete Math. 106/107 (1992) 199–207 (a collection of contributions in honour of Jack van Lint).
[11] T. Ericson, V. Zinoviev, Spherical codes generated by binary partitions of symmetric pointsets, IEEE Trans. Inform. Theory 41 (1) (1995) 107–129.
[12] T. Ericson, V. Zinoviev, Codes on Euclidean Spheres, Vol. 63, North-Holland Mathematical Library, Amsterdam, 2001.
[13] Z. Füredi, M. Ruszinkó, An improved upper bound of the rate of Euclidean superimposed codes, IEEE Trans. Inform. Theory 45 (2) (1999) 799–802.
[14] V.I. Levenshtein, Packing of polynomial metric spaces, in: Proceedings of the Third International Workshop on Information Theory, Sochi, 1987, pp. 271–274.
[15] V.I. Levenshtein, Designs as maximum codes in polynomial metric spaces, Acta Appl. Math. 25 (1992) 1–83.
[16] N.J.A. Sloane, How to pack lines, planes, 3-spaces, etc. [webpage], http://www.research.att.com/∼njas/grass/.
[17] J.H. van Lint, J.J. Seidel, Equilateral point sets in elliptic geometry, Proc. Kon. Nederl. Akad. Wet. A 69 (1966) 335–348.